(V) BY SIMILAR CALCULATIONS

$P(T_5 < \infty) = 1/2$ AND

GIVEN $T_5 < \infty$ THE # OF TIMES $X_n = 5$ IS A GEOMETRIC$\left(\frac{1}{6}\right)$

(i.e WITH EXPECTATION $1/_{1/6} = 6$)

SO THE EXPECTED TIME SPENT IN 5 IS $\frac{1}{2}\, 6 = 3$

4)Customers arrive at a shop according to a Poisson process of rate λ. Each customer is a male with probability $\frac{1}{3}$ and a female with probability $\frac{2}{3}$ independently of other custiomers gender or arrival times. Each customer will buy something at the store with probability $\frac{1}{2}$ independently of all the previous information and of other customers purchasing decisions.

(i)What is the probability nothing is bought by a customer arriving in time $[0,1]$?

(ii) If in $[0,1]$ exactly 4 men arrived who made purchases, what is the conditional probability that during this time exactly 3 women came into this store.

(iii) Suppose now that on $[0,\infty)$ arriving women stay an exponential time with parameter c_F, while men stay exponential time with parameter c_H (again all these exponmential variables are independent of each other and arival times). Let F_t and H_t be the number of women (respectively men) in the shop at time t. For $V_t = (F_t, H_t)$ write down the Q matrix.

(iv) For which values of λ and c_F is the jump chain for F_t transient?

(i) THE PROCESS OF ARRIVALS FOR CUSTOMERS WHO WILL EVENTUALLY BUY SOMETHING IS $\lambda \times \frac{1}{2}$ ($= \lambda/2$.) POISSON PROCESS

SO PROB = PROB NO ARRIVALS IN $[0,1]$ FOR THIS PROCESS

$= e^{-\lambda/2}$.

(ii) THE PROCESS OF MALE ARRIVALS & WOMEN ARRIVALS ARE INDEP POISSON PROCESSES OF RATES $\lambda/3$ $2\lambda/3$ RESP.

SO ANSWER = PROB (EXACTLY 3 ARRIVALS IN $[0,1]$ FOR $2\lambda/3$ P.P)

$$= \frac{e^{-2\lambda/3}(2\lambda/3)^3}{3!}$$

(iii) $Q_{(m,n),(m',n')} = 0$ UNLESS $|m-m'|+|n-n'| \leq 1$

$Q_{(m,n),(m+1,n)} = 2\lambda/3$ $\quad Q_{(m,n)(m-1,n)} = mc_F$

$Q_{(m,n),(m,n+1)} = \lambda/3$ $\quad Q_{(m,n)(m,n-1)} = nc_H$

SO $Q_{(m,n)(m,n)} = -(\lambda + mc_F + nc_H)$

(iv) FOR (F_t) THE JUMP CHAIN IS FOR $i > 0$

$$P_{i,i+1} = \frac{2\lambda/3}{2\lambda/3 + i c_F}$$

$$P_{i,i-1} = \frac{i c_F}{2\lambda/3 + i c_F}$$

SO IF $c_F > 0$ THEN $\exists\, i_0$ SO THAT

$$\forall i \geq i_0 \quad P_{i,i+1} < 1/10$$
$$P_{i,i-1} > 9/10$$

SO BY COMPARISON TO B&D PROCESS

$$P^{i_0}(T_{i_0-1} < \infty) = 1$$

$\Rightarrow$ CHAIN IS RECURRENT $\forall c_F > 0$

5)Consider the Markov chain on $S = \{0, 1 \cdots 6\}$ with transition matrix below

$$\begin{pmatrix} 0 & 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/2 & 0 & 1/6 & 0 & 1/6 & 1/6 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

a) What are the communicating classes ?
b) What approximately is $P^{100}_{0\,3}$?
c) What approximately is $P^{100}_{0\,6}$?

$0 \rightleftarrows 1$ BUT $0 \to 3$ BUT $3 \not\to 0$
$0 \to 6$ BUT $6 \not\to 0$

AND $3 \to 4 \to 2 \to 3$ $3 \not\to 5, 6$
$5 \to 6 \to 5$ $5 \not\to \{0,1,2,3,4,5\}$

SO COMMUNICATING CLASSES ARE $\{0,1\}$ (TRANSIENT)
$\{2,3,4\}$ RECURRENT.
$\{5,6\}$

(b) CONSIDER MC ON $\{2,3,4\}$ HAS MATRIX $\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

SO $\pi(3) = \frac{1}{3}\pi(2)$
$\pi(2) = \frac{1}{3}\pi(2) + \frac{1}{2}\pi(4)$
$\Rightarrow \pi(2) = \frac{3}{4}\pi(4)$
$\cancel{\pi(4) = \frac{1}{3}\pi(2) + \pi(3) + \frac{1}{2}\pi(4)}$

$1 = \pi(2) + \pi(3) + \pi(4) = \pi(2) + \frac{1}{3}\pi(2) + \frac{4}{3}\pi(2) = \frac{8}{3}\pi(2)$

$\Rightarrow \pi = (\cancel{\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{4}{8}})$ $3/8$ $1/8$ $4/8$ AND AS THIS CHAIN (i.e. ON $\{2,3,4\}$) IS APERIODIC $P^{100}_{03} \approx P_0(T_3 < \infty)\ 1/8$ IS $h_i = P^i(T_3 < \infty)$

$h_0 = \frac{1}{3}h_1 + 1/3$ $h_1 = \frac{1}{2}h_0 + 1/3$ $\Rightarrow$ $h_0 - h_1 = \frac{1}{3}h_1 - \frac{1}{2}h_0$
$\Rightarrow h_0 = \frac{4/3}{3/2}h_1 = \frac{8}{9}h_1$

$\Rightarrow h_0 = \frac{3}{8}h_0 + 1/3 \Rightarrow h_0 = \frac{8}{15}$ $P^{100}_{03} \approx 8/15 \times \frac{1}{8} \approx 1/15$

(iii) THIS IS MORE COMPLICATED AS CHAIN ON $\{5,6\}$

HAS TRANSITION MATRIX $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ AND IS PERIODIC

FORTUNATELY THE CHAIN ON $\{0,1\}$ ALTERNATES

(STARTING AT 0) BETWEEN 0 (FOR EVEN TIMES) AND

1 FOR ODD TIMES UNTIL IT HITS $\{2,3,4\}$ OR $\{5,6\}$

TO HIT 6 CAN ONLY COME FROM 0 (FOR FIRST HITTING OF $\{5,6\}$)

OR TO HIT 5. 1

SO HITTING $\{5,6\}$ FROM STARTING AT 0 CAN ONLY OCCUR

BY HITTING 6 ON ODD TIME OR HITTING 5 ON EVEN TIME

$\Rightarrow \quad P^{100}_{0,6} = 0.$